Published by Institute of Physics Publishing for SISSA/ISAS
Received: *December 10, 2002*
Accepted: *January 21, 2003*

Four-point functions in $N = 4$ SYM

Paul J. Heslop
Department of Physics, Martin Luther Universität
Halle, Germany
E-mail: heslop@physik.uni-halle.de

Paul S. Howe
Department of Mathematics, King's College
London, UK
E-mail: phowe@mth.kcl.ac.uk

JHEP01(2003)043

Abstract: A new derivation is given of four-point functions of charge Q chiral primary multiplets in $N = 4$ supersymmetric Yang-Mills theory. A compact formula, valid for arbitrary Q, is given which is manifestly superconformal and analytic in the internal bosonic coordinates of analytic superspace. This formula allows one to determine the spacetime four-point function of any four component fields in the multiplets in terms of the four-point function of the leading chiral primary fields. The leading term is expressed in terms of $1/2Q(Q-1)$ functions of two conformal invariants and a number of single variable functions. Crossing symmetry reduces the number of independent functions, while the OPE implies that the single-variable functions arise from protected operators and should therefore take their free form. This is the partial non-renormalisation property of such four-point functions which can be viewed as a consequence of the OPE and the non-renormalisation of three-point functions of protected operators.

Keywords: AdS-CFT and dS-CFT Correspondence, Extended Supersymmetry, Superspaces.

Contents

1. Introduction

Four-dimensional superconformal theory has attracted renewed interest in recent years mainly due to the Maldacena conjecture which, in particular, relates IIB string theory on $AdS_5 \times S^5$ to $N = 4$ Yang-Mills theory [1]. In order to make explicit checks of this conjecture it is necessary to try to make computations on the field theory side of the correlation functions. The non-renormalisation theorems for two- and three-point functions of protected operators have been studied by many authors (see [2] for reviews and references) and have now been established non-perturbatively for all protected operators [3].[1] It is therefore of interest to study four-point functions, particularly those of the chiral primary operators which correspond to the Kaluza-Klein states of IIB supergravity. There has been a good deal of work on this topic too, as we review briefly below. In this paper we discuss a new approach to these correlation functions which is valid for arbitrary charges.

The main tool that we shall use is harmonic superspace, which was introduced in a seminal paper in 1984 [4]; superspaces with additional bosonic coordinates were also discussed in a slightly different approach, commonly referred to as projective superspace, in [5]. The maximally supersymmetric Yang-Mills theory in four spacetime dimensions can be formulated in a rather neat way in $N = 4$ analytic superspace [6, 7], an example of a harmonic superspace. This space has 8 even and 8 odd coordinates, compared to ordinary $N = 4$ superspace which has 4 even and 16 odd coordinates. The additional even coordinates describe a complex compact manifold which is a coset space of the internal symmetry group SU(4), while the reduction in the number of odd coordinates can be viewed as a type of generalised chirality. In this setting, the free $N = 4$ SYM field strength superfield is a single-component holomorphic field W. The interacting field strength is covariantly analytic and so is not itself a field on this space, but gauge-invariant combinations of it are; in particular, the so-called chiral primary superfields $A_Q := \mathrm{tr}(W^Q)$ are holomorphic fields on analytic superspace (the gauge group is assumed to be $\mathrm{SU}(N_c)$). This fact was observed some time ago [7, 8], and a programme of investigating correlation functions of such operators was initiated [8, 9]. In particular, a lot of effort was expended on the analysis of the four-point function of four CPOs, mainly in $N = 2$ analytic superspace where gauge-invariant products of the hypermultiplet are analytic superfields. Such four-point functions can be written in terms of a prefactor multiplied by a function of the superconformal invariants, and it was hoped that analyticity in the internal coordinates would lead to restrictions on this function since the natural superinvariants one can write down are all rational functions of the coordinates. This is indeed the case, and the full analysis was carried out in [10]. It is important to note that all of the unitary irreducible representations in $N = 4$ can be realised as superfields on analytic superspace, although, in general, these fields carry superindices [11, 12]. This shows that the analytic superspace formalism is complete, and this is essential for the study of OPEs and correlation functions as we shall see in this paper .

The basic result of [10] is that the $N = 2$ four-point function of charge 2 operators can be solved in terms of a single arbitrary function of the two independent spacetime cross-ratios together with two other functions which satisfy two coupled partial differential equations. These can be solved in terms of two functions of single variables which can be constructed from the cross-ratios [13]. Explicit calculations in perturbation theory [14, 15, 16] and in the instanton sector [17] have been used to find particular examples of the two-variable function, and these calculations also showed that the additional single-variable functions take their free-theory forms.

[1] It should be noted, however, that there are operators with vanishing anomalous dimensions which may not have non-renormalised three-point functions.

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A study of the four-point function of stress-tensor multiplets in $N = 4$ was made in [13]. This is related to the charge 2 $N = 2$ correlator because $N = 4$ SYM splits into a vector multiplet and a hypermultiplet in the $N = 2$ formulation of the theory. With the use of crossing symmetry and the reduction formula it was shown that this four-point correlator is determined by a single function of two-variables together with a part which is free[2], and moreover agrees with the supergravity result [19, 20, 21]. This is referred to as partial non-renormalisation of the four-point function. This result has recently been confirmed in a component field OPE calculation where it was shown that the single-variable functions arise from the interchange of protected operators [22]. The four-point function has also been studied in perturbation theory [23, 16], in terms of instanton contributions [23, 24] and using the OPE [25]–[28].

One implication of this result, first noted in [25], is that some series A operators are protected from developing anomalous dimensions even though it had been widely thought that they would be unprotected. This was confirmed in a field theory calculation of three-point functions of two CPOs and a third operator in [29]; it can also be seen from instanton contributions [23]. Subsequently it was shown that this phenomenon has a simple explanation and that there are many more operators of this type: the series A operators which saturate unitarity bounds and which can be written in $N = 4$ analytic superspace in terms of derivatives and CPOs will be guaranteed to satisfy the same shortening conditions as in the free theory and will furthermore not be descendants [30]. Hence such operators will be protected provided that the superconformal Ward identities are satisfied.

In this paper we revisit four-point functions of CPOs in the $N = 4$ analytic superspace formalism. The key idea is to express the function of invariants in terms of the Schur functions of the group GL(2|2). The group GL(2|2) arises naturally in the analytic superspace formalism as part of the isotropy subgroup of the superconformal group which defines analytic superspace as a coset (the analogous group in Minkowski space is the spin group SL(2)). This group acts by the adjoint action on a $(2|2) \times (2|2)$ matrix Z which is constructed from the four coordinate matrices. The Schur functions are $\mathrm{str}\,\mathcal{R}(Z)$ where $\mathcal{R}$ denotes a finite-dimensional representation of GL(2|2). Although we do not prove it here, this approach is related to the OPE because one can identify the representations $\mathcal{R}$ that occur with the intermediate operators in the OPE (both primary and descendants). The possible representations are restricted by analyticity, and divide into two classes, the short, or atypical representations and the long, or typical, representations. The Schur functions are straightforward to compute, and combining this information with crossing symmetry one arrives at compact expressions for the four-point functions of four identical charge Q CPOs in terms of a number of functions of two conformal invariants and a number of functions of only one variable. We emphasise the fact that the formulae we give are valid for the entire multiplets and not just for the leading scalar field correlation functions. In principle one can compute the four-point function of any four component fields in these multiplets in a systematic fashion from the superfield formulae given in the paper.

[2]The reduction formula relates the derivative of an n-point function with respect to the coupling to an $(n+1)$-point function with one insertion of the integrated on-shell action. It was first used in the current context in [18]

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The functions of one variable are associated with the short representations and the latter correspond to protected operators in the OPE. Since the contribution of an intermediate operator is related to the three-point function of two CPOs and the operator in question, and since such three-point functions are non-renormalised when the operator is protected, it follows that the contribution of these operators should take the same form as in the free theory. This is the partial non-renormalisation theorem; from this point of view it can be seen as a corollary of the existence of protected operators and the non-renormalisation of three-point functions of protected operators.

In section 2 we review the analytic superspace formalism for $N=4$ SYM and then give a brief discussion of four-point functions in $N=4$. In particular, we state the main result for the contribution of long operators to such correlators. In section 3 we discuss Schur polynomials for various (super)groups of interest. In section 4 we use the Schur functions to solve the Ward identities for four-point functions of scalar fields or CPO supermultiplets in four-dimensional theories with $N=0,2,4$ supersymmetries. These results are for four operators of the same type but not necessarily identical. In section 5 we specialise to the case of identical operators by imposing crossing symmetry. We give formulae for the number of independent functions of two variables which arise for different values of Q in $N=2$ and $N=4$. Finally, in section 6 we discuss free four-point functions for four charge 2 multiplets in both $N=2$ and $N=4$ and show how these can be written in terms of functions of a single variable of the type that arise in the preceding analysis.

2. Four-point functions

The field strength supermultiplet of $N=4$ super Yang-Mills theory consists of six scalar fields transforming under the real six-dimensional representation of SU(4), four chiral spinor fields and their conjugates and field strength tensor of the spin one gauge field. In the free theory these fields can be packaged into a single-component field $W(X)$ defined on analytic superspace. This superspace has half the number of odd coordinates as ordinary Minkowski superspace but also has an additional compact internal space which can be viewed as the Grassmannian of two-planes in $\mathbb{C}^4$ or as the coset space $S(\mathrm{U}(2)\times\mathrm{U}(2))\backslash\mathrm{SU}(4)$. Locally this space is the same as complexified spacetime. The local coordinates are

$$X^{AA'}=\begin{pmatrix} x^{\alpha\alpha'} & \lambda^{\alpha a'} \\ \pi^{a\alpha'} & y^{aa'}\end{pmatrix} \tag{2.1}$$

where the xs are the spacetime coordinates, the ys are local coordinates on the internal space and the λs and πs are the odd coordinates. An infinitesimal superconformal transformation acts on this space by

$$\delta X^{AA'}=(\mathcal{V}X)^{AA'}=B^{AA'}+A^{A}{}_{B}X^{BA'}+X^{AB'}D_{B'}{}^{A'}+X^{AB'}C_{B'B}X^{BA'} \tag{2.2}$$

where $\mathcal{V}$ is the vector field generating the transformation and where A,B,C and D are all $(2|2)\times(2|2)$ supermatrices. The field W transforms by

$$\delta W=\mathcal{V}W+\Delta W \tag{2.3}$$

where

$$\Delta := \mathrm{str}(A + XC) \tag{2.4}$$

As it stands, this defines a $\mathfrak{pgl}(4|4)$ transformation. The Lie superalgebra $\mathfrak{psl}(4|4)$ is the complexification of the $N = 4$ superconformal algebra and can be obtained by imposing the constraint $\mathrm{str}(A + XC) = \mathrm{str}(D + CX)$, while $\mathfrak{pgl}(4|4)$ extends $\mathfrak{psl}(4|4)$ by an abelian factor corresponding to the group $\mathrm{U}(1)_Y$. It as first suggested in [18] that $\mathrm{U}(1)_Y$ could be a symmetry of the supercurrent four-point function and this was confirmed in [31]. In the following analysis we shall use the bigger group.

The chiral primary operators are defined to be single traces of powers of W. In the interacting theory (we assume the gauge group is $\mathrm{SU}(N_c)$), W is covariantly analytic and is not actually a field on analytic superspace. However, gauge-invariant products of the field strength are analytic fields, and so we can define the CPOs as

$$A_Q := \mathrm{tr}(W^Q) \tag{2.5}$$

These operators transform in a similar way to W but with a term $Q\Delta A_Q$.

The four-point functions we are interested in are

$$\langle A_Q(X_1)A_Q(X_2)A_Q(X_3)A_Q(X_4)\rangle := \langle QQQQ\rangle \tag{2.6}$$

The superconformal Ward identities are

$$\sum_{i=1}^{4} (\mathcal{V}_i + Q\Delta_i)\,\langle QQQQ\rangle = 0 \tag{2.7}$$

We can reduce the problem of solving these identities to that of finding the superconformal invariants by introducing the propagator g_{12} which is the two-point function of Ws in the free theory:

$$\langle W(1)W(2)\rangle \sim g_{12} = \mathrm{sdet}(X_{12}^{-1}) = \frac{\hat{y}_{12}^2}{x_{12}^2} \tag{2.8}$$

where $X_{ij} := X_i - X_j$ and where $\hat{y} = y - \pi x^{-1}\lambda$. Here we use the convention that inverse coordinates have lower indices, e.g. $(x^{-1})_{\alpha'\alpha}$. With the aid of the propagator we can write

$$\langle QQQQ\rangle = (g_{12}g_{34})^Q \times F_Q \tag{2.9}$$

where F_Q is a function of the superinvariants. A crucial property of the four-point function is that it is analytic in the internal coordinates (the ys). This must be so since each A_Q is a polynomial in y of degree $2Q$.

In previous analyses that have been made of the four-point function in analytic superspace (mainly in $N = 2$) the function F has been written in terms of an explicit basis of invariants and then the analyticity properties of functions of these has been studied. In the current paper we adopt a different approach to this problem by writing the invariants in terms of the Schur polynomials of a certain variable Z.

Consider the problem of finding an invariant function F of n points X_i in analytic superspace. It is clear that translation invariance requires F to be a function of difference variables X_{ij}. Now define, for any three points $i \neq j \neq k$,

$$X_{ijk} := X_{ij} X_{jk}^{-1} X_{ki} \tag{2.10}$$

where no summations are implied. These transform by

$$\delta X_{ijk} = A_i X_{ijk} + X_{ijk} D_i \tag{2.11}$$

where $A_i := A + X_i C$ and $D_i := D + C X_i$. Provided that F is invariant under A and D transformations, invariance under C-transformations will follow if F is a function of $(n-2)$ variables of the type of X_{ijk}. We now define four-point variables by

$$X_{ijkl} := X_{ij} X_{jk}^{-1} X_{kl} X_{li}^{-1} \tag{2.12}$$

where all points must be different. These variables transform by

$$\delta X_{ijkl} = A_i X_{ijkl} - X_{ijkl} A_i \tag{2.13}$$

As we shall show in more detail elsewhere, the n-point invariants can be constructed from $(n-3)$ variables of this type all of which transform under the adjoint representation of $\mathfrak{gl}(2|2)$ at point 1.

For four points there is only one independent such variable which we shall call Z,

$$Z := X_{2134} = X_{21} X_{13}^{-1} X_{34} X_{42}^{-1} \tag{2.14}$$

The problem of four-point invariants is thus equivalent to finding all functions of the matrix Z which are invariant under the adjoint action of $\mathfrak{gl}(2|2)$. The finite version of this is invariance under the adjoint action of the group $\mathrm{GL}(2|2)$,

$$Z \mapsto G^{-1} Z G \tag{2.15}$$

where $G \in \mathrm{GL}(2|2)$. (Note that the full group does not act on Z, only $\mathrm{PGL}(2|2)$.) It can be shown that full invariants of this type actually correspond to superconformal invariants which are also invariant under $\mathrm{U}(1)_Y$; for the case in hand, all four-point invariants have this property. For $n \geq 5$ points one can show that there are invariants which are invariant only under $\mathrm{SL}(2|2)$ rather than $\mathrm{GL}(2|2)$ which give rise to superconformal invariants which are not invariant under $\mathrm{U}(1)_Y$.

The Schur polynomials are functions of the form $\mathrm{str}(\mathcal{R}(Z)) := S_{\mathcal{R}}(Z)$ where $\mathcal{R}$ denotes a finite dimensional representation of $\mathrm{GL}(2|2)$. Any such representation can be described by a $\mathrm{GL}(2|2)$ Young tableau labeled by four integers $\langle m_1, m_2, m_3, m_4 \rangle$, but it is possible to restrict the types of tableau that can occur if we also allow powers of the superdeterminant of Z. We thus arrive at the formula

$$\langle QQQQ \rangle = (g_{12} g_{34})^Q \sum_{p,\mathcal{R}} C_{p,\mathcal{R}} \, (\mathrm{sdet}\, Z)^p S_{\mathcal{R}}(Z) \tag{2.16}$$

This formula is related to the OPE approach to four-point functions. The OPE for two A_Qs was given in [3]; it reads

$$A_Q(1)A_Q(2) \sim \sum_{\mathcal{R}} (g_{12})^{Q-\frac{q}{2}} \mathcal{R}(X_{12}) \mathcal{O}^q_{\mathcal{R}\mathcal{R}}(2) + \cdots \tag{2.17}$$

Here, the dots denote the contributions of the descendants of the primary fields $\mathcal{O}^q_{\mathcal{R}\mathcal{R}}$ and $\mathcal{R}(X_{12})$ means a product of X_{12} with both the primed and the unprimed indices projected into the representation $\mathcal{R}$. Each primary field in this expansion has charge $q = L - (J_1 + J_2)$ where L is the dilation weight and J_1, J_2 are the two spin quantum numbers of the superconformal representation under which the operator transforms. In general, an operator on analytic superspace will be a tensor (or quasi-tensor) field carrying I A-type superindices and an equal number of primed superindices and will transform under finite-dimensional irreducible representations of the two GL(2|2) groups which act on the primed and unprimed superindices A, A'. The number of indices of each type must be equal in order to have vanishing R-weight, and all of the indices must be covariant (subscript) in order for the representation to be unitary. Since the two operators on the left-hand-side of the OPE are scalars, operators contributing to the right-hand-side will have $\mathcal{R} = \mathcal{R}'$. Analyticity places restrictions on the representations that can appear in the OPE, but these can be seen more easily directly from analyticity of the four-point function. A key restriction imposed by unitarity is that one only sees representations $\mathcal{R}$ corresponding to covariant tensorial representations. The conjugate representations to these will therefore not appear in the expansion of the four-point function.

The contribution of an operator $\mathcal{O}^q_{\mathcal{R}\mathcal{R}}$ to the four-point function $\langle QQQQ\rangle$ has the form

$$\langle QQQQ\rangle \sim \frac{(A_{QQ\mathcal{O}})^2}{A_{\mathcal{O}\mathcal{O}}} (g_{12}g_{34})^Q (\operatorname{sdet} Z)^{q/2} \sum_{\mathcal{R}'} C_{\mathcal{R}'} S_{\mathcal{R}'}(Z) \tag{2.18}$$

where $\mathcal{R}'$ denotes a representation with a Young tableau which contains the Young tableau of $\mathcal{R}$, $A_{QQ\mathcal{O}}$ is the coefficient of the 3-point function $\langle A_Q A_Q \mathcal{O}\rangle$, $A_{\mathcal{O}\mathcal{O}}$ is the coefficient of the 2-point function $\langle \mathcal{O}\mathcal{O}\rangle$, and $C_{\mathcal{R}'}$ are purely numerical constants (in particular $C_{\mathcal{R}} = 1$.)

It is simple to obtain the restrictions that analyticity imposes on F_Q. From the form of Z it is apparent that there can only be poles in the (13) and (24) channels from each $S_{\mathcal{R}}(Z)$. Looking at the (12) channel we have $g_{12}^Q \sim (y_{12}^2)^Q$ while $\operatorname{sdet}(Z)^p \sim (y_{12}^2)^{-p}$. Hence analyticity in this channel requires $p \leq Q$. The same result is obtained from the (34) channel. For the (13) channel $\operatorname{sdet} Z^p \sim (y_{13}^2)^p$ while the leading singularity from $S_{\mathcal{R}}(Z)$ will arise from the term with most factors of y. As we shall see below, the tableaux we need to consider are those which have two rows of arbitrary length (provided that the second is not longer than the first) together with a single first column of length r or the trivial representation which has $r = 0$. The worst singularity one can have for all of these possibilities is $(y_{13}^2)^{-r}$ from which we learn that $p \geq r$. The same result is obtained from the (24) channel. So the restrictions due to analyticity are simply

$$r \leq p \leq Q\,. \tag{2.19}$$

In order to write the correlator in a more explicit form it is useful to use the G transformation of Z to bring it diagonal form. We can thus write it in terms of its eigenvalues as

$$Z \sim \mathrm{diag}(X_1, X_2|Y_1, Y_2) \tag{2.20}$$

In section (3) we shall give explicit formulae for the Schur polynomials in terms of these variables, but for the moment we shall simply state that they can be used to show that the contribution of the long operators to the correlator, can be written in the form

$$\langle QQQQ\rangle_2 = (g_{13}g_{24})^Q SF^Q(X_1, X_2, Y_1Y_2) \tag{2.21}$$

where S is a universal function of the eigenvalues which will be given explicitly later on. F^Q is a polynomial in the variables $(Y_1 + Y_2)$ and Y_1Y_2 of degree $Q - 2$ with coefficients which depend on the variables X_1, X_2. Now these variables, evaluated at zero in the odd coordinates, will simply be the eigenvalues of the GL(2) matrix $z = x_{21}x_{13}^{-1}x_{34}x_{42}^{-1}$ which occurs in the bosonic problem, so that these coefficient functions can equivalently be regarded as functions of the two independent four-point conformal invariants in spacetime. At this stage, therefore, (2.21) tells us that the four-point function of CPOs is determined completely (to all orders in odd coordinates) by $1/2\ Q(Q - 1)$ functions of two variables, together with a part which describes the contribution of the protected operators. However, we can also impose crossing symmetry as we have taken the CPOs to be identical. Under the interchange of point 1 and 3 $Z \mapsto (1 - Z)$, while the prefactor $(g_{13}g_{24})^Q S$ is invariant. Under the interchange of points 1 and 4 $Z \mapsto Z^{-1}$ while the prefactor changes to itself times $(Y_1Y_2)^{Q-2}(X_1X_2)^{-Q-2}$. Demanding symmetry under these two operations is sufficient to ensure full crossing symmetry and reduces the number of independent functions considerably. For low values of Q, we find that the number of independent functions is 1 for $Q = 2$ and for $Q = 3$, 2 for $Q = 4$, 3 for $Q = 5$ and so on. We shall discuss crossing symmetry in more detail in section 5 where we give a formula for the number of independent functions for arbitrary Q.

To summarise, the four-point function of four identical CPOs of charge Q is given by (2.21) together with the contribution from protected operators. The function F^Q is determined by a number of functions of two variables which depends on Q. This result generalises the partial non-renormalisation theorem for the supercurrent multiplet (which corresponds to $Q = 2$). The partial non-renormalisation concerns the contribution of the protected operators via the OPE. As we shall see below, these operators give rise to functions of a single variable (i.e. one of the eigenvalues X_1 or X_2). Since these functions are related to the three-point functions of two A_Qs and the protected operator in question, it follows that they are non-renormalised. We thus expect these single-variable functions to take the same form in the interacting theory as they would in the free theory.

3. Schur polynomials

As we have seen above a four-point invariant function F can be written entirely as a function of the $(2|n) \times (2|n)$ matrix $Z = X_{21}X_{13}^{-1}X_{34}X_{42}^{-1}$ which is invariant under the adjoint action, $Z \to G^{-1}ZG$ for any GL(2|2) matrix G. In fact we can consider the more general case of GL($2|n$), since then $n=0$ corresponds to Minkowski space, $n=1$ to $N=2$ analytic superspace and $n=2$ to $N=4$ analytic superspace.

Such an invariant function is known as a class function. It is useful to write such functions in terms of the Schur polynomials or characters:

$$S_{\mathcal{R}}(Z) := \operatorname{str} \mathcal{R}(Z)\,. \tag{3.1}$$

where $\mathcal{R}$ is any representation of G. Indeed, for Lie groups, the Peter-Weyl theorem states that the characters of irreducible representations span a dense subspace of the space of continuous class functions (see [32]).

In practice the Schur polynomials can be obtained as follows. The representation $\mathcal{R}$ is specified by a Young tableau with, say m boxes. The expression $\mathcal{R}(Z)$ is obtained by taking the tensor product of m Zs and symmetrising the upstairs indices according to the Young tableau. The supertrace over this representation is then obtained by contracting the upstairs and downstairs indices, and dividing by the hook length formula for the Young tableau in question.[3] So for example for the fundamental representation, $\mathcal{R} = \square$ one has simply the usual supertrace of Z,

$$\mathcal{R} = \square \Rightarrow S_{\mathcal{R}}(Z) = \operatorname{str}(Z)\,, \tag{3.2}$$

whereas for the symmetric representation one has

$$\mathcal{R} = \square\square \Rightarrow S_{\mathcal{R}}(Z) = \frac{1}{2}\left(Z^A{}_A Z^B{}_B (-1)^{A+B} + Z^B{}_A Z^A{}_B (-1)^B\right) = \frac{1}{2}\left(\operatorname{str}(Z)^2 + \operatorname{str}(Z^2)\right). \tag{3.3}$$

since the Hook-length formula gives 2 in this case. We shall expand the invariant functions we are interested in in terms of these Schur polynomials. The assumption that this is possible can be shown to be correct for four-point functions which have a double OPE expansion interpretation.

3.1 Schur polynomials in terms of invariant variables

A useful way to write the Schur functions explicitly is in terms of the eigenvalues of Z. To do this one chooses a matrix G which diagonalises Z so that

$$Z = \operatorname{diag}(X_1, X_2|Y_1, \ldots, Y_n)\,. \tag{3.4}$$

Since the 4-point invariant function F is invariant under the adjoint action, it can be written entirely as a function of the $2+n$ eigenvalues of Z,

$$F = F(X_1, X_2, Y_1 \ldots, Y_n)\,. \tag{3.5}$$

[3]The hook length formula is the product of the hook lengths of all the boxes in a Young tableau, the hook length of a box being the number of squares below and to the right of the box, including the box itself once.

It is straightforward, in principle, to find these variables although, in practice, it is in general a hard problem to find the matrix G which diagonalises Z. In addition, because there are matrices G which can interchange X_1, X_2 and also (separately) permute the Ys, the function F must be a symmetric function of X_1, X_2 and of the Y_is.

To make things more explicit we now express the Schur polynomials in terms of the eigenvalues of Z. We consider several different cases in turn.

3.1.1 GL(1)

First, as a warm up, let us consider the trivial case GL(1) in which case Z is simply a complex number. Representations of GL(1) are given by Young tableaux with only one row with n boxes say. In this case one simply finds that

$$S_{\mathcal{R}}(Z) = Z^n \tag{3.6}$$

One must also consider conjugate representations, obtained by replacing Z by Z^{-1}. These can be accommodated by allowing n to take take negative values in the above formula.

3.1.2 GL(2)

This corresponds to ordinary four-dimensional conformal symmetry (i.e. $N = 0$). In this case we shall write the diagonal form of Z as $Z = \mathrm{diag}(x_1, x_2)$ and so the Schur polynomials can be written in terms of these two variables. Representations of GL(2) are given by Young tableaux with two rows of lengths $a+b$ and b say, and the corresponding Schur polynomials are given as:

$$\mathcal{R}_{ab} = \overbrace{\begin{array}{|c|c|c|c|}\hline \ & \ & \cdots & \ \\ \hline \ & \ & \cdots & \ \\ \hline\end{array}}^{b}\overbrace{\begin{array}{|c|c|c|}\hline \ & \cdots & \ \\ \hline\end{array}}^{a} \quad \Longrightarrow S_{ab}(Z) = (x_1 x_2)^b \left(\frac{x_1^{a+1} - x_2^{a+1}}{x_1 - x_2} \right) \tag{3.7}$$

Again, one should also consider conjugate representations. However, GL(2) representations are equivalent to their conjugate representations up to multiplication of a determinant. In fact, all representations can be included by allowing b to take negative values in the above formula. However, an OPE analysis together with CFT unitary bounds shows that the conjugate representations are not needed in the four-point function. Note that since the conformal group is non-compact b may also take non-integer values corresponding to the presence of operators with anomalous dimensions.

We briefly note here that the variables x_1, x_2 are precisely the same as the variables x, z used in [22]; they are related to the usual cross-ratios u, v by (3.36).

We shall want to consider arbitrary linear combinations of the Schur polynomials; these can be rewritten in terms of a function of two variables as follows:

$$\sum_{a,b} C_{ab} S_{ab}(Z) = G(x_1, x_2) \tag{3.8}$$

where

$$G(x_1, x_2) := \sum_{a,b} C_{ab}\ (x_1 x_2)^b \frac{x_1^{a+1} - x_2^{a+1}}{x_1 - x_2}. \tag{3.9}$$

and where C_{ab} are arbitrary constants.

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3.1.3 GL(2|1)

The case GL(2|1) corresponds to $N = 2$ superconformal symmetry. The easiest way to find the Schur polynomials for supergroups is to consider first the representations as representations of the maximal bosonic subgroup, in this case $\mathrm{GL}(2|1) \supset \mathrm{GL}(2) \times \mathrm{GL}(1)$. Splitting $\mathrm{GL}(M|N)$ representations into $\mathrm{GL}(M) \times \mathrm{GL}(N)$ is done in a similar way as for $\mathrm{GL}(M+N)$ except that one considers the conjugate representation of $\mathrm{GL}(N)$ (for more detail on this see [33].)

All representations of SL(2|1) can be given by Young tableaux with two rows, and hence representations of GL(2|1) can be given by Young tableaux with two rows up to multiplication by the superdeterminant. The representations come in two types, R_{ab} and R_a, which have Young tableaux and $\mathfrak{sl}(2|1)$ Dynkin labels as follows:

$$\mathcal{R}_{ab} = \overbrace{\begin{smallmatrix}\square\square & \cdots & \square\square\\ \square\square & \cdots & \square\square\end{smallmatrix}}^{b-1}\overbrace{\square\square\cdots\square\square}^{a} \qquad [a, a+b] \qquad b > 1,\ a \geq 0 \tag{3.10}$$

$$\mathcal{R}_{a} = \overbrace{\square\square\cdots\square\square}^{a+1} \qquad \begin{cases} [a, a+1] & a \geq 0 \\ [0,0] & a = -1 \end{cases} \tag{3.11}$$

The $\mathcal{R}_{ab}$s are long (or typical) representations whereas the $\mathcal{R}_a$s are short (or atypical). Note that a must be an integer while b can be non-integral. In the context of superconformal field theory b must be real and greater than 1 and the presence of non-integral representations corresponds directly to anomalous dimensions. We remark that the interpretation of a Young diagram is the same as in the bosonic case with regard to symmetrisation, except for the fact that symmetry or anti-symmetry is understood to be generalised. Thus, for example, $\mathcal{R}_1$ corresponds to a tensor which is generalised symmetric on two $\mathbb{C}^{2|1}$ vector indices.

Under $\mathrm{GL}(2|1) \supset \mathrm{GL}(2) \times \mathrm{GL}(1)$ we find

$$\begin{aligned} \mathcal{R}_a &= \overbrace{\square\square\cdots\square\square}^{a+1} \to (\overbrace{\square\square\cdots\square\square}^{a+1}, 1) + (\overbrace{\square\square\cdots\square\square}^{a}, \square) & (3.12)\\ \mathcal{R}_{02} &= \begin{smallmatrix}\square\\ \square\end{smallmatrix} \to (\begin{smallmatrix}\square\\ \square\end{smallmatrix}, 1) + (\square, \square) + (1, \square\square)\,. & (3.13) \end{aligned}$$

The Schur polynomials are just the supertraces over these representations of $Z \sim \mathrm{diag}(X_1, X_2|Y)$, so we can write the Schur polynomials of the supergroup in terms of the corresponding Schur polynomials of the maximal bosonic subgroup $\mathrm{GL}(2) \times \mathrm{GL}(1)$, in terms of the variables X_1, X_2 for GL(2) and Y for GL(1). So for example, corresponding to the above two representations we have the Schur polynomials

$$\begin{aligned} S_a(Z) &= \frac{X_1^{a+2} - X_2^{a+2}}{X_1 - X_2} - Y\frac{(X_1^{a+1} - X_2^{a+1})}{X_1 - X_2} \qquad a \geq -1 & (3.14)\\ S_{02}(Z) &= X_1X_2 - Y(X_1 + X_2) + Y^2\,. & (3.15) \end{aligned}$$

Notice that a minus sign occurs whenever there are an odd number of Y's. This is because of the minus sign in the definition of the supertrace.

To find the Schur polynomials for more general representations with two rows, notice that under GL(2|1) $\supset$ GL(2) $\times$ GL(1)

$$\mathcal{R}_{ab} = \overbrace{\begin{smallmatrix}\square\square & \cdots & \square\square \\ \square\square & \cdots & \square\square\end{smallmatrix}}^{b-1}\overbrace{\begin{smallmatrix}\square & \cdots & \square\square\end{smallmatrix}}^{a} = \left\{ \left(\begin{smallmatrix}\square\\ \square\end{smallmatrix}, 1\right) + (\square, \square) + (1, \square\square) \right\} \times \left(\overbrace{\begin{smallmatrix}\square\square & \cdots & \square\square \\ \square\square & \cdots & \square\square\end{smallmatrix}}^{b-2}\overbrace{\begin{smallmatrix}\square & \cdots & \square\square\end{smallmatrix}}^{a}, 1 \right) \tag{3.16}$$

and since the Schur polynomials respect the multiplication of representations (i.e. $S_{\mathcal{R}}S_{\mathcal{R}'} = S_{\mathcal{R}\mathcal{R}'}$) we find that

$$S_{ab}(Z) = (X_1X_2)^{b-2}\left(\frac{X_1^{a+1} - X_2^{a+1}}{X_1 - X_2}\right) S_{02}(Z)\,. \tag{3.17}$$

As in the purely bosonic case we will consider linear combinations of Schur polynomials and rewrite them in terms of functions as follows

$$\sum_a C_aS_a = \frac{X_1f(X_1) - X_2f(X_2)}{X_1 - X_2} - Y\left(\frac{f(X_1) - f(X_2)}{X_1 - X_2}\right) := \mathcal{F}[f] \tag{3.18}$$

where

$$f(X) = \sum_a C_aX^{a+1} \tag{3.19}$$

and

$$\sum_{a,b} C_{ab}S_{ab} = G(X_1, X_2)(X_1X_2 - Y(X_1 + X_2) + Y^2) := \mathcal{G}[G] \tag{3.20}$$

where

$$G(X_1, X_2) := \sum_{a,b} C_{ab}\ (X_1X_2)^{b-2}\ \frac{X_1^{a+1} - X_2^{a+1}}{X_1 - X_2}. \tag{3.21}$$

We see that the short representations lead to a function of one variable, $f(X)$ whereas the long representations lead to a function of two variables $G(X_1, X_2)$. These are the origin of the one and two variable functions obtained in the analysis of 4-point functions as we shall see shortly.

The long representations are a priori only valid for $b > 1$ although b can take non-integer values corresponding to the presence of anomalous dimensions and the related notion of quasi-tensors [3]). However, we may consider the formula for S_{ab} for the special case $b = 1$, S_{a1}. According to (3.10) this appears to correspond to the representation with Dynkin labels $[a, a+1]$ in other words $\mathcal{R}_{a1}$ has the same Dynkin labels as $\mathcal{R}_a$. However, these two are not the same as can be seen by comparing the Schur polynomials S_{a1} and S_a. The reason for this apparent discrepancy is that $\mathcal{R}_{a1}$ is a reducible representation where as $\mathcal{R}_a$ is irreducible. This can be seen from the corresponding Schur polynomials which satisfy

$$S_{a1}(Z) = S_{a-1}(Z) - \mathrm{sdet}(Z)^{-1}S_a(Z). \tag{3.22}$$

So $\mathcal{R}_{a1}$ contains the representations R_a and R_{a-1}.[4]

[4]This result is directly related to the reducibility of certain superconformal operators at the threshold of the unitary bounds, such as the Konishi multiplet.

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Note that the $N = 2$ superconformally invariant variables X_1, X_2 are related to the conformally invariant variables x_1, x_2 by $X_1 = x_1 + O(\theta), X_2 = x_2 + O(\theta)$ where θ are Grassmann odd coordinates.

In principle, we should also include the conjugate representations which in the supersymmetric case are not equivalent to non-conjugate representations. However, an analysis of the OPE in conjunction with the superconformal unitary bounds shows that the conjugate representations are not needed for 4-point functions.

3.1.4 GL(2|2)

The GL(2|2) case, corresponding to $N = 4$ superconformal symmetry, is similar to the GL(2|1) case. We shall illustrate the procedure for finding Schur polynomials with two examples, and then just write them down for general representations.

In this case there are again two types of representation which we denote $\mathcal{R}_{abc}$, $\mathcal{R}_{ac}$ with Young tableaux and Dynkin labels

$$\mathcal{R}_{abc} = \begin{array}{l} \overbrace{\begin{smallmatrix}\square\square & \cdots & \square\square \\ \square\square & \cdots & \square\square\end{smallmatrix}}^{b}\overbrace{\square\square\cdots\square\square}^{a} \\ c\left\{\begin{array}{l}\square \\ \vdots \\ \square\end{array}\right. \end{array} \qquad [a, a+b, c], \;\; b > 1, \;\; a, c \geq 0 \tag{3.23}$$

$$\mathcal{R}_{ac} = \begin{array}{l} \overbrace{\square\square\cdots\square\square}^{a+2} \\ c\left\{\begin{array}{l}\square \\ \vdots \\ \square\end{array}\right. \end{array} \qquad \begin{array}{l} [a, a+1, c] \;\; a, c \geq 0 \\ [0, 0, c+1] \;\; a = -1, c \geq 0 \end{array} \tag{3.24}$$

The trivial representation must be treated seperately in this case and we shall call it $\mathcal{R}_0$. For these representations a and c are integral while b can again be non-integral. As in the GL(2|1) case, to find the Schur polynomials for these representations, we begin by splitting the supergroup into its maximal bosonic subgroup. For example, under $\mathrm{GL}(2|2) \supset \mathrm{GL}(2) \times \mathrm{GL}(2)$

$$\mathcal{R}_{a0} = \overbrace{\square\square\cdots\square\square}^{a+2} \to (\overbrace{\square\square\cdots\square\square}^{a+2}, 1) + (\overbrace{\square\square\cdots\square\square}^{a+1}, \square) + (\overbrace{\square\square\cdots\square\square}^{a}, \square\square) \tag{3.25}$$

$$\mathcal{R}_{020} = \begin{smallmatrix}\square\square\\\square\square\end{smallmatrix} \to (\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}, 1) + (\begin{smallmatrix}\square\square\\\square\end{smallmatrix}, \square) + (\begin{smallmatrix}\square\\\square\end{smallmatrix}, \square\square) + (\square\square, \begin{smallmatrix}\square\\\square\end{smallmatrix}) + (\square, \begin{smallmatrix}\square\square\\\square\end{smallmatrix}) + (1, \begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}) \tag{3.26}$$

and the corresponding Schur polynomials, given in terms of $Z \sim \mathrm{diag}(X_1, X_2, Y_1, Y_2)$ are

$$\begin{aligned} S_{a0}(Z) &= \frac{X_1^{a+3} - X_2^{a+3}}{X_1 - X_2} - (Y_1 + Y_2)\frac{X_1^{a+2} - X_2^{a+2}}{X_1 - X_2} + Y_1 Y_2 \frac{X_1^{a+1} - X_2^{a+1}}{X_1 - X_2} \\ a &\geq -1 \end{aligned} \tag{3.27}$$

$$\begin{aligned} S_{020}(Z) &= (X_1 X_2)^2 - X_1 X_2 (X_1 + X_2)(Y_1 + Y_2) + Y_1 Y_2 \left(\frac{X_1^3 - X_2^3}{X_1 - X_2}\right) + \\ &\quad + \left(\frac{Y_1^3 - Y_2^3}{Y_1 - Y_2}\right) X_1 X_2 - Y_1 Y_2 (Y_1 + Y_2)(X_1 + X_2) + (Y_1 Y_2)^2 . \end{aligned} \tag{3.28}$$

Note that this time both the space-time group and the internal group is GL(2).

Note also that the $N = 4$ variables X_1, X_2 are not of course the same as in the $N = 2$ case. It should be clear from the context whether we are talking about $N = 4$ variables or $N = 2$ variables.

By decomposing the representations of GL(2|2) into representations of GL(2) × GL(2) in this way one finds the following Schur polynomials:

$$S_{abc} = (-1)^c (X_1X_2)^{b-2} \left(\frac{X_1^{a+1} - X_2^{a+1}}{X_1 - X_2}\right) \left(\frac{Y_1^{c+1} - Y_2^{c+1}}{Y_1 - Y_2}\right) \times S_{020} \tag{3.29}$$

$$\begin{aligned} S_{a,c} = {} & (-1)^{c+1} (X_1X_2) \left(\frac{X_1^{a+2} - X_2^{a+2}}{X_1 - X_2}\right) \left(\frac{Y_1^{c} - Y_2^{c}}{Y_1 - Y_2}\right) + \\ & + (-1)^c (X_1 + X_2) \left(\frac{X_1^{a+2} - X_2^{a+2}}{X_1 - X_2}\right) \left(\frac{Y_1^{c+1} - Y_2^{c+1}}{Y_1 - Y_2}\right) + \\ & + (-1)^c (X_1X_2) \left(\frac{X_1^{a+1} - X_2^{a+1}}{X_1 - X_2}\right) (Y_1Y_2) \left(\frac{Y_1^{c-1} - Y_2^{c-1}}{Y_1 - Y_2}\right) + \\ & + (-1)^{c+1} \left(\frac{X_1^{a+2} - X_2^{a+2}}{X_1 - X_2}\right) \left(\frac{Y_1^{c+1} - Y_2^{c+1}}{Y_1 - Y_2}\right) (Y_1 + Y_2) + \\ & + (-1)^{c+1} (X_1X_2) \left(\frac{X_1^{a} - X_2^{a}}{X_1 - X_2}\right) (Y_1Y_2) \left(\frac{Y_1^{c} - Y_2^{c}}{Y_1 - Y_2}\right) + \\ & + (-1)^c \left(\frac{X_1^{a+1} - X_2^{a+1}}{X_1 - X_2}\right) (Y_1Y_2) \left(\frac{Y_1^{c+1} - Y_2^{c+1}}{Y_1 - Y_2}\right). \\ & (c \geq 0\,, \quad a \geq -1) \end{aligned} \tag{3.30}$$

If we analytically continue S_{ac} down to the case $a = -2,\ c = 0$, which should correspond to the trivial representation, we obtain instead $S_{-2,0} = 1 - Y_1Y_2/X_1X_2$, whereas for the trivial representation we should obtain the answer $S_0 = 1$, so we treat this case separately.

We will again consider linear combinations of these Schur polynomials. These can be expressed in terms of functions of one or two variables:

$$\sum_{a,b} C_{abc} S_{abc}(Z) = (-1)^c G_c(X_1, X_2) \left(\frac{Y_1^{c+1} - Y_2^{c+1}}{Y_1 - Y_2}\right) \times S_{020}(Z) := \mathcal{G}_c[G_c] \tag{3.31}$$

where

$$G_c(X_1, X_2) = \sum_{a,b} C_{abc} (X_1X_2)^{b-2}\, \frac{X_1^{a+1} - X_2^{a+1}}{X_1 - X_2} \tag{3.32}$$

and

$$\begin{aligned} \sum_a C_{ac} S_{ac} = {} & (-1)^{c+1} (X_1X_2) \left(\frac{X_1^2 f_c(X_1) - X_2^2 f_c(X_2)}{X_1 - X_2}\right) \left(\frac{Y_1^{c} - Y_2^{c}}{Y_1 - Y_2}\right) + \\ & + (-1)^c (X_1 + X_2) \left(\frac{X_1^2 f_c(X_1) - X_2^2 f_c(X_2)}{X_1 - X_2}\right) \left(\frac{Y_1^{c+1} - Y_2^{c+1}}{Y_1 - Y_2}\right) + \\ & + (-1)^c (X_1X_2) \left(\frac{X_1 f_c(X_1) - X_2 f_c(X_2)}{X_1 - X_2}\right) (Y_1Y_2) \left(\frac{Y_1^{c-1} - Y_2^{c-1}}{Y_1 - Y_2}\right) + \\ & + (-1)^{c+1} \left(\frac{X_1^2 f_c(X_1) - X_2^2 f_c(X_2)}{X_1 - X_2}\right) \left(\frac{Y_1^{c+1} - Y_2^{c+1}}{Y_1 - Y_2}\right) (Y_1 + Y_2) + \end{aligned}$$

$$+(-1)^{c+1}(X_1X_2)\left(\frac{f_c(X_1)-f_c(X_2)}{X_1-X_2}\right)(Y_1Y_2)\left(\frac{Y_1^c-Y_2^c}{Y_1-Y_2}\right)+$$

$$+(-1)^c\left(\frac{X_1f_c(X_1)-X_2f_c(X_2)}{X_1-X_2}\right)(Y_1Y_2)\left(\frac{Y_1^{c+1}-Y_2^{c+1}}{Y_1-Y_2}\right):=\mathcal{F}_c[f_c] \quad (3.33)$$

where

$$f_c(X_1)=\sum C_{ac}X_1^a\,. \quad (3.34)$$

Once again we see that the short representations of GL(2|2) lead to functions of one variable, whereas the long representations lead to functions of two variables.

Formula (3.29) is a priori only valid for $b>1$. However, as in the $N=2$ case it is also possible to use the formula for $b=1$, in which case it corresponds to a reducible representation. In fact one finds that

$$S_{a1c}(Z)=S_{(a-1)(c+1)}(Z)+\mathrm{sdet}(Z)^{-1}S_{ac}(Z)\,. \quad (3.35)$$

giving the analogue of (3.22) for $N=2$.

For the same reason as in the $N=2$ case we do not need to include the conjugate representations of GL(2|2).

3.2 Relation to other variables

We may now ask how these relate these variables to those used previously in various papers. In [22] it was found useful to consider the variables x_1,x_2 (called x,z in the paper) where

$$x_1x_2=u:=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}\qquad(1-x_1)(1-x_2)=v:=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}\,. \quad (3.36)$$

Interestingly enough, the eigenvalues of Z which we are using, x,z, are precisely these variables in [22]. To see this note that $u=\det(Z)=xz$ and $1-Z=X_{23}X_{31}^{-1}X_{14}X_{42}^{-1}$ which means that $v=\det(1-Z)=(1-x)(1-z)$ giving exactly the relations (3.36). This elucidates the importance of the variables x,z. (We have called these (x_1,x_2) above.)

For $n=1$ (i.e. N=2 superconformal field theory) the invariant variables

$$V=\frac{T+U-1}{1+S-T}\qquad S'=SV\qquad T'=T(1+V) \quad (3.37)$$

where

$$S=\frac{\mathrm{sdet}\,X_{14}\ \mathrm{sdet}\,X_{23}}{\mathrm{sdet}\,X_{12}\ \mathrm{sdet}\,X_{34}}\qquad T=\frac{\mathrm{sdet}\,X_{13}\ \mathrm{sdet}\,X_{24}}{\mathrm{sdet}\,X_{12}\ \mathrm{sdet}\,X_{34}}\qquad U=\mathrm{str}(X_{12}^{-1}X_{23}X_{34}^{-1}X_{41}) \quad (3.38)$$

were found to be useful in [10]. These can be straightforwardly related to the eigenvalues of Z, i.e. X_1,X_2,Y. It is perhaps more illuminating to consider instead the matrix $Z'=X_{23}X_{34}^{-1}X_{41}X_{12}^{-1}$ (this is straightforwardly related to Z by permuting the insertion points) and its corresponding eigenvalues X_1',X_1',Y'. Then one finds straightforwardly that

$$S=-\frac{X_1'X_2'}{Y'}\qquad T=\frac{(1-X_1')(1-X_2')}{(1-Y')}\qquad U=X_1'+X_2'-Y' \quad (3.39)$$

and that

$$S'=X_1'X_2'\qquad T'=(1-X_1')(1-X_2')\qquad V=-Y' \quad (3.40)$$

Thus we see that V is (up to a minus sign) the same as the internal eigenvalue Y whereas S',T' have a similar relationship to X_1',X_2' as u,v do to x,z (compare with equation (3.36).)

4. Construction of four-point functions

In this section we consider the four-point functions of various gauge invariant operators in (super)conformal field theories with the aid of the representation theory of $\mathrm{GL}(2|n)$. Consideration of the OPE expansion (2.16-2.18) tells us that the four-point function of charge Q scalar operators A_Q can be expanded in the form

$$\langle QQQQ\rangle = (g_{12}g_{34})^Q \sum_{p,\mathcal{R}} C_{p,\mathcal{R}}\, \mathrm{sdet}(Z)^p\, S_{\mathcal{R}}(Z) \tag{4.1}$$

where $p \geq 0$, $C_{p,\mathcal{R}}$ are arbitrary constants and $S_{\mathcal{R}}(Z)$ are Schur polynomials with positive powers of the eigenvalues only (i.e. we do not need conjugate representations).

Before considering analytic superspace itself, we consider two simpler cases for comparison, namely $\mathrm{GL}(2)$, which corresponds to the component formalism for conformal field theories, and $\mathrm{GL}(0|n)$ which gives some insight into the internal $\mathrm{SU}(2n)$ symmetry of superconformal field theory.

4.1 Four-point functions in $N = 0$

We consider the four-point function of scalar operators, ϕ_Q, with dilation weight Q. From (4.1) this can be expanded in the form

$$\langle \phi_Q\phi_Q\phi_Q\phi_Q\rangle = g_{12}^Q g_{34}^Q \sum_{\mathcal{R}} C_{\mathcal{R}} S_{\mathcal{R}}(Z) \tag{4.2}$$

where we have omitted the factor $\det^p$ because this can be absorbed into the expression for the Schur polynomial (since $\det^p(Z)S_{ab}(Z) = S_{a,b+p}(Z)$). Using (3.8) we can rewrite this in terms of a function of two variables:

$$\langle \phi_Q\phi_Q\phi_Q\phi_Q\rangle = g_{12}^Q g_{34}^Q G(x_1,x_2) \tag{4.3}$$

where

$$G(x_1,x_2) := \sum_{a=0}^{\infty}\sum_b C_{ab}(x_1x_2)^b\, \frac{x_1^{a+1}-x_2^{a+1}}{x_1-x_2} \tag{4.4}$$

and where we have written $C_{\mathcal{R}_{ab}} = C_{ab}$. Note that the sum for b will be over some countable set of real numbers, corresponding to the values of b for which C_{ab} is non-zero.

4.2 Space-time independent four-point functions

It is interesting to consider this formalism for the supergroup $\mathrm{GL}(0|n)$. This is equivalent to considering the four-point function of spacetime independent $\mathrm{SU}(2n)$ tensors. The operator A_Q is equivalent to a spacetime independent tensor carrying the $\mathrm{SU}(2n)$ representation with Dynkin labels $[0,\dots,0,Q,0,\dots,0]$ where there are $n-1$ 0s on each side of the Q.

First consider $n = 1$ in which case $Z = Y$. A_Q is equivalent to a space-time independent tensor carrying the $Q+1$ dimensional representation of $\mathrm{SU}(2)$. For example, if $Q = 1$, then $A_Q(y) = A_1 + A_2 y$ is equivalent to the tensor A_i.

We can again use (4.1) to write the four-point function as

$$\langle QQQQ\rangle = g_{12}^Q g_{34}^Q \sum_p C_p \, Y^{-p} \,. \tag{4.5}$$

Here we have used the fact that in this case $S_{\mathcal{R}_p}(Z) = \det^p(Z) = \mathrm{sdet}(Z)^{-p}$ where $\mathcal{R}_p$ s given by a Young tableau with one row of p boxes. The latter identity is simply the definition of superdeterminant for the group $\mathrm{GL}(0|n)$. We therefore omit $S_{\mathcal{R}}(Z)$ but allow p to take positive and negative values.

Now A_Q is analytic in the variables Y_i and so the right hand side must be also. Using the fact that $Y = y_{12}y_{23}^{-1}y_{34}y_{41}^{-1}$, we find that the powers of y_{ij} and corresponding conditions are

$$\begin{aligned} y_{12}^Q y_{12}^{-p} &\Rightarrow p \leq Q \\ y_{13}^p &\Rightarrow p \geq 0 \end{aligned} \tag{4.6}$$

and so there are $Q+1$ terms. This is precisely what one gets by contracting SU(2) indices.

We now consider $n=2$ corresponding to SU(4) representations. A field A_Q is equivalent to an SU(4) tensor with Dynkin labels $[0, Q, 0]$. If we consider the four-point function of four of these we have

$$\langle QQQQ\rangle = (g_{12}g_{34})^Q \sum_{\mathcal{R}} C_{\mathcal{R}} \quad S_{\mathcal{R}}(Z) \,. \tag{4.7}$$

Here we omit the term $\mathrm{sdet}(Z)^p$ and instead consider GL(2) Schur polynomials (3.7)

$$S_{cd}(Z) = (Y_1Y_2)^{-d}\left(\frac{{Y_1}^{c+1} - {Y_2}^{c+1}}{Y_1 - Y_2}\right) \tag{4.8}$$

with positive and negative values for d. Here we have let $Z \sim \mathrm{diag}(Y_1, Y_2)$. Now $Z = y_{21}y_{13}^{-1}y_{34}y_{42}^{-1}$ and $Y_1Y_2 = \mathrm{sdet}\, Z^{-1} = (y_{12}^2 y_{34}^2)(y_{13}^2 y_{24}^2)^{-1}$ so that

$$(Y_1Y_2)^{-d} = (y_{12}^2 y_{34}^2)^{-d}(y_{13}^2 y_{24}^2)^d \tag{4.9}$$

On the other hand, the second factor in the sum is always a positive power of Z. So the only poles here will come from inverting y_{13} and y_{42}. Thus this factor will contain a factor of $(y_{13}^2 y_{24}^2)^{-c}$.

Analyticity in the variables y_{12} or y_{34} therefore implies that $Q-d \geq 0$ while analyticity in y_{13} or y_{24} implies that $d - c \geq 0$. So we conclude that $0 \leq c \leq d \leq Q$. This gives a total of $\frac{1}{2}(Q+1)(Q+2)$ terms (eg. 6 terms if $Q = 2$ corresponding to the four point function of four tensors in the 20' representation of SU(4)). This is precisely the number of different irreducible representations one obtains in $[0, Q, 0] \times [0, Q, 0]$ which is what one would expect. The formalism therefore provides a nice way to study the internal group which has some relevance to the leading term of the four-point functions of CPOs as we shall see later. However, the real power of the formalism occurs when one combines the spacetime and internal parts.

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4.3 Four-point function in $N = 2$

4.4 $\langle 2222 \rangle$

We now consider the four-point function of four CPOs of charge two in $N = 2$ SCFT, reproducing results in [28] (note, however, that we write down the complete four-point function for four charge 2 multiplets, whereas previously results have been only for the first term in theta.) We expand this in the form (4.1):

$$\langle 2222 \rangle = g_{12}^2 g_{34}^2 \sum_{p,\mathcal{R}} C_{p,\mathcal{R}}\, \mathrm{sdet}(Z)^p\ S_{\mathcal{R}}(Z) \tag{4.10}$$

where $g_{12} = \hat{y}_{12}/x_{12}^2$ with $\hat{y}_{12} = y_{12} - \pi_{12} x_{12}^{-1} \lambda_{12}$. We can restrict the $\mathcal{R}$s to be given by Young tableaux with only two rows since a tableau with more than two rows can be replaced by one with two rows if we include an appropriate power of $\mathrm{sdet}(Z)$.)

We now consider the constraints imposed by analyticity. In the (12) channel we have a factor $(y_{12})^{-p}$ coming from the superdeterminant and a factor $(y_{12})^2$ from the propagator, so that analyticity in y_{12} implies that $p \leq 2$. The same result is obtained from the (34) channel . The $S_{\mathcal{R}}$ factor will only contribute possible singularities in the (13) and (24) channels. From the symmetry properties of the representation $\mathcal{R}$ it is clear that we can have no more than r factors of y_{13}^{-1} (or y_{24}^{-1}), where r is the number of rows in the tableau. In the (13) channel we have a factor of y_{13}^p from the superdeterminant, so analyticity in this channel implies that $p \geq r$. The same result is obtained for the channel (24). Since $r = 0, 1, 2$ is the number of rows of a representation we see that $p = 2$ for long representations $\mathcal{R}_{ab}$, $p = 2$ or 1 for the representations $\mathcal{R}_a$ and $p = 0, 1, 2$ for the trivial representation.

We can therefore write

$$\langle 2222 \rangle = g_{13}^2 g_{24}^2 \left(\sum_{a=0}^{\infty} \sum_{b>1} C_{ab}\ S_{ab}(Z) + \sum_{a=-1}^{\infty} C_a S_a(Z) + \frac{Y}{X_1 X_2} \sum_{a=-2}^{\infty} C'_a S_a(Z) \right) \tag{4.11}$$

where we have absorbed the factor $(\mathrm{sdet}\, Z)^2$ into the prefactor and where we have defined $C_{ab} := C_{2,\mathcal{R}_{ab}}$, $C_a := C_{2,\mathcal{R}_a}$, $C'_a := C_{1,\mathcal{R}_a}$ and $C'_{-2} = C_{0,\mathcal{R}_0}$. We have also used the fact that if we put $a = -2$ into the formula (3.14) for S_a we find $S_{-2} = \frac{Y}{X_1 X_2}$. Note that by the sum over $b > 1$ we mean that the sum is over some countable set of real numbers greater than one. In the free theory, the sum will be over integers as illustrated in section 6, but in the interacting theory the sum will in general be over non-integers corresponding to the presence of anomalous dimensions.

Using (3.18-3.20) we can write this expression in terms of two functions of one variable $f(X_1), f'(X_1)$ and a function of two variables $G(X_1, X_2)$

$$\langle 2222 \rangle = g_{13}^2 g_{24}^2 \left(\mathcal{G}[G] + \mathcal{F}[f] + \frac{Y}{X_1 X_2} \mathcal{F}[f'] \right) \tag{4.12}$$

where

$$f(X_1) = \sum_{a=-1}^{\infty} C_a X_1^{a+1} \qquad\qquad f'(X_1) = \sum_{a=-2}^{\infty} C'_a X_1^{a+1}$$

$$G(X_1,X_2) = \sum_{a=0}^{\infty}\sum_{b>1} C_{ab}(X_1X_2)^{b-2}\ \frac{X_1^{a+1}-X_2^{a+1}}{X_1-X_2}\,. \tag{4.13}$$

We therefore arrive at a compact formula for the fully supersymmetric $N=2$ four-point function of four charge 2 operators in terms of two functions of one variable f, f' and a function of two variables G.

There are two functions of one variable since there are two sequences of short representations in the four-point function (corresponding to $p=1$ and $p=2$) and there is one function of two variables since there is only one sequence of long representations. This is a general feature, for each sequence of short representations one gets a function of one variable, and for each sequence of long representations one gets a function of two variables.

The two-variable contribution to the correlator can be written, using (3.20) in the form

$$\langle 2222\rangle \sim (g_{12}g_{34})^2 S_{02}\ G(X_1,X_2) \tag{4.14}$$

where we recall that $S_{02} = X_1X_2 - Y(X_1+X_2) + Y^2$.

One can, if one prefers, absorb one of the single variable functions into the function of two variables, G by using (3.22) and by allowing terms with $b=1$ in G. Equation (3.22) implies that

$$\frac{Y}{X_1X_2}\mathcal{F}[h(X_1)] - \mathcal{F}\left[\frac{h(X_1)}{X_1}\right] + \mathcal{G}\left[\frac{h(X_1)-h(X_2)}{(X_1X_2)(X_1-X_2)}\right] = 0 \tag{4.15}$$

for any function of one variable h, and since the functionals $\mathcal{F}, \mathcal{G}$ are linear, the expression for the four point function (4.12) is invariant under

$$f' \to f' - h \qquad f \to f + \frac{h}{X_1} \qquad G \to G - \frac{h(X_1)-h(X_2)}{X_1X_2(X_1-X_2)} \tag{4.16}$$

So in particular, with the choice $h=f'$ we can absorb f' entirely into $G(X_1,X_2)$ and write the entire four-point function in terms of a single function of one variable and a function of two variables at the expense of allowing lower powers of X_1, X_2 in G.

From (2.18) we can see that the coefficients C_a, C'_a are related to the three-point functions of two Ts and a protected operator. In [3] we showed that all such three-point functions receive no corrections to their free-field values. Therefore one expects the functions f, f' to take their free-field values.

4.5 $\langle QQQQ\rangle$

It is straightforward to extend this to the case of CPOs of arbitrary charge Q.

$$\langle QQQQ\rangle = (g_{12}g_{34})^Q \sum_{p,\mathcal{R}} C_{p,\mathcal{R}}\ \mathrm{sdet}(Z)^{-p}\ S_{\mathcal{R}}(Z)\,. \tag{4.17}$$

Analyticity implies that $r \le p \le Q$, where r is the number of rows in the Young tableau which can be at most two.

We have

$$\langle QQQQ\rangle = (g_{13}g_{24})^Q \Bigg(\sum_{p=2}^{Q}\sum_{a=0}^{\infty}\sum_{b>1}\left(\frac{Y}{X_1X_2}\right)^{Q-p} C_{pab}S_{ab}(Z) + \\ + \sum_{p=1}^{Q}\sum_{a=-1}^{\infty}\left(\frac{Y}{X_1X_2}\right)^{Q-p} C_{pa}S_a(Z) + C\left(\frac{Y}{X_1X_2}\right)^{Q}\Bigg). \qquad (4.18)$$

It is convenient to rewrite the last term as

$$C\left(\frac{Y}{X_1X_2}\right)^{Q} = \left(\frac{Y}{X_1X_2}\right)^{Q-1} C_{1,-2}S_{-2}(Z)\,. \qquad (4.19)$$

This four-point function can then be rewritten in terms of $Q-1$ functions of two variables, $G_p(X_1,X_2)$ and Q functions of of one variable, $f_p(X_1)$, as

$$\langle QQQQ\rangle = (g_{13}g_{24})^Q \left(\sum_{p=2}^{Q}\left(\frac{Y}{X_1X_2}\right)^{Q-p}\mathcal{G}[G_p] + \sum_{p=1}^{Q}\left(\frac{Y}{X_1X_2}\right)^{Q-p}\mathcal{F}[f_p]\right) \qquad (4.20)$$

where

$$f_p = \sum_{a=-1}^{\infty} C_{pa}X_1^{a+1} \qquad p=2,\dots,Q \qquad (4.21)$$

$$f_1 = \sum_{a=-2}^{\infty} C_{1a}X_1^{a+1} \qquad (4.22)$$

$$G_p(X_1,X_2) = \sum_{a=0}^{\infty}\sum_{b>1} C_{pab}(X_1X_2)^{b-2}\,\frac{X_1^{a+1}-X_2^{a+1}}{X_1-X_2}. \qquad (4.23)$$

Using (3.20) we can rewrite the two-variable contribution in the form

$$\langle QQQQ\rangle_2 = (g_{13}g_{24})^Q S_{02}\sum_{p=2}^{Q} G_p(X_1,X_2)Y^{Q-p} \qquad (4.24)$$

where we have absorbed some factors of X_1 and X_2 into the $Q-1$ G_ps.

As in the $\langle 2222\rangle$ case we could use (3.22) to remove some of the one-variable functions, f_p by subsuming them into the two variable functions. By repeated application of (4.15) one obtains the relation

$$\left(\frac{Y}{X_1X_2}\right)^{p}\mathcal{F}[h(X_1)] = \mathcal{F}\left[\frac{h(X_1)}{X_1^p}\right] - \mathcal{G}\left[\frac{h'(X_1)-h'(X_2)}{(X_1X_2)(X_1-X_2)}\right] \qquad (4.25)$$

where

$$h'(X_1) := \left(1+\frac{1}{X_1}+\cdots+\frac{1}{X_1^{p-1}}\right)h(X_1) \qquad (4.26)$$

which can be used to absorb the functions f_p, $p=1,\dots,Q-1$ into f_Q and G_p. This will leave us with only one function of one variable, but at the expense of allowing powers of X_1^{-p} in the remaining functions f_Q and G_p.

Since the functions of one variable will take their free theory values (that is, they are non-renormalised) it does not seem to be particularly advantageous to absorb the single variable functions in this way.

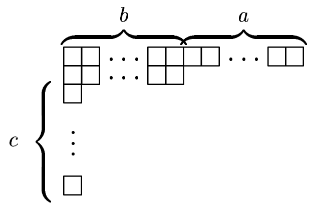
4.6 Four-point functions in $N = 4$

4.6.1 Charge 2

We can make a similar construction in $N = 4$ for the four-point function of four CPOs. We begin with the simplest case of charge Q. For this case the operator A_2 is the supercurrent supermultiplet T. We first expand the four point function as

$$\langle TTTT \rangle = (g_{12} g_{34})^2 \sum_{p,\mathcal{R}} C_{p,\mathcal{R}} \;\; \mathrm{sdet}(Z)^p \; S_{\mathcal{R}}(Z) \tag{4.27}$$

and then consider analyticity in the various internal coordinates. Since we are now in $N = 4$ the representations $\mathcal{R}$ that occur in the above formula are representations of GL(2|2) . The most general such representation (up to multiplication by a superdeterminant) which can occur can be specified by a Young tableau of the form



(4.28)

As in the previous examples we need only look at channels (12) and (13) in order to obtain the restrictions due to analyticity. The superdeterminant contributes a factor $(y_{12}^2)^{-p}(y_{13}^2)^p$ so that analyticity in the (12) channel implies that $p \leq 2$. The maximum number of powers of y_{13}^{-1} that can arise from the representation $\mathcal{R}$ described by the above tableau will occur when all the boxes in the first two columns are filled with y_{13}^{-1}s. There are $r - c$ boxes with two y_{13}^{-1}s giving a factor $(y_{13}^2)^{c-r}$ and c boxes with y_{13}^{-1}s in a symmetrised combination. This gives a factor of $(y_{13}^{-1})^c$ and this in turn gives rise to a factor of $(y_{13}^2)^{-c}$. Analyticity in the (13) channel therefore yields the constraint $p \geq r$. Here r is the total number of rows in the tableau, so $p = 2$ for representations with two rows, $p = 1$ or 2 for representations $\mathcal{R}_{a,0}$ with only 1 row, and $p = 0, 1, 2$ for the trivial representation $\mathcal{R}_0$. So the four-point function has the form

$$\begin{aligned}\langle TTTT \rangle \; = \; g_{13}^2 g_{24}^2 \Bigg(& \sum_{a=0}^{\infty} \sum_{b>1} C_{2ab0} \;\; S_{ab0}(Z) + \sum_{a=-1}^{\infty} C_{2a1} S_{a1}(Z) + \sum_{a=-2}^{\infty} C_{2a0} S_{a0}(Z) + \\ & + \left(\frac{Y_1 Y_2}{X_1 X_2} \right) \sum_{a=-2}^{\infty} C_{1a0} S_{a0}(Z) + C \Bigg) .\end{aligned} \tag{4.29}$$

In this expression we have used the fact that $S_{-2,0} = 1 - Y_1 Y_2 / X_1 X_2 = S_0 - (Y_1 Y_2 / X_1 X_2) S_0$ to reexpress some terms. As in previous cases, the sum over b is a sum over some countable set of real numbers.

This can then be written as

$$\langle TTTT \rangle = g_{13}^2 g_{24}^2 \left(\mathcal{G}_{ab0}[G_0] + \mathcal{F}_1[f] + \mathcal{F}_0[g] + \left(\frac{Y_1 Y_2}{X_1 X_2} \right) \mathcal{F}_0[h] + C \right) \tag{4.30}$$

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with

$$G_0(X_1,X_2) = \sum_{a=0}^{\infty}\sum_{b>1} C_{2ab0}(X_1X_2)^{b-2}\ \frac{X_1^{a+1}-X_2^{a+1}}{X_1-X_2} \tag{4.31}$$

$$f(X_1) = \sum_{a=-1}^{\infty} C_{2a1}X_1^a \tag{4.32}$$

$$g(X_1) = \sum_{a=-2}^{\infty} C_{2a0}X_1^a \tag{4.33}$$

$$h(X_1) = \sum_{a=-2}^{\infty} C_{1a0}X_1^a\,. \tag{4.34}$$

We have rewritten the four point function in terms of one function of two variables, $G(X_1,X_2)$, three functions of one variable, $f(X_1)$, $g(X_1)$, $h(X_1)$ and a constant, C. As in the $N=2$ case one can eliminate a function of one variable using (3.35) by including terms of order $1/(X_1X_2)$ in $G(X_1,X_2)$.

4.7 $\langle QQQQ\rangle$

The above formula can easily be generalised to the four-point function of charge Q CPOs in analytic superspace

$$\langle QQQQ\rangle = (g_{12}g_{34})^Q \sum_{p,\mathcal{R}} C_{p,\mathcal{R}}\ \operatorname{sdet}(Z)^p\ \operatorname{str}\mathcal{R}(Z)\,. \tag{4.35}$$

The analyticity conditions become $p\leq Q$ (from the (12) channel) and $p\geq r$ (from the (13) channel), so that the maximum number of rows that is allowed is now Q.

We therefore obtain

$$\begin{aligned}\langle QQQQ\rangle &= g_{13}^Q g_{24}^Q\Bigg(\sum_{p=2}^{Q}\sum_{c=0}^{p-2}\sum_{a=0}^{\infty}\sum_{b>1}\left(\frac{Y_1Y_2}{X_1X_2}\right)^{Q-p} C_{pabc}S_{abc} + \\ &\qquad + \sum_{p=1}^{Q}\sum_{c=0}^{p-1}\sum_{a=-2}^{\infty}\sum_{b>1}\left(\frac{Y_1Y_2}{X_1X_2}\right)^{Q-p} C_{pac}S_{ac} + C\Bigg) \\ &= g_{13}^Q g_{24}^Q\left(\sum_{p=2}^{Q}\sum_{c=0}^{p-2}\left(\frac{Y_1Y_2}{X_1X_2}\right)^{Q-p}\mathcal{G}_c[G_{pc}] + \sum_{p=1}^{Q}\sum_{c=0}^{p-1}\left(\frac{Y_1Y_2}{X_1X_2}\right)^{Q-p}\mathcal{F}_c[f_{pc}] + C\right)\end{aligned} \tag{4.36}$$

We see that the four-point function can be expressed in terms of $(1/2)Q(Q-1)$ functions of two variables, G_{pc}, $(1/2)Q(Q+1)$ functions of one variable, f_{pc} and a constant, C.

We can once again use (3.35) to absorb $(1/2)Q(Q-1)$ of the one-variable functions into the two variable functions to leave us with Q functions of one variable, $(1/2)Q(Q-1)$ functions of two variables and a constant.

We can rewrite the two-variable contribution in a slightly more explicit form by using (3.31). We find

$$\langle QQQQ\rangle_2 = (g_{13}g_{24})^Q S_{020}\sum_{p=2}^{Q}\sum_{c=0}^{p-2} G_{pc}\left(\frac{Y_1^{c+1}-Y_2^{c+1}}{Y_1-Y_2}\right)(Y_1Y_2)^{Q-p} \tag{4.37}$$

where S_{020} is a universal factor which is given in equation (3.28) and where we have again absorbed some explicit factors of $X_1$and X_2 into the functions G_{pc}. Comparing this result with equation (2.21) in section 2, we see that

$$F^Q = \sum_{p=2}^{Q} \sum_{c=0}^{p-2} G_{pc} \left(\frac{Y_1^{c+1} - Y_2^{c+1}}{Y_1 - Y_2} \right) (Y_1 Y_2)^{Q-p} \tag{4.38}$$

5. Crossing symmetry

In this section we shall discuss the restrictions that crossing symmetry imposes on the four-point functions. We begin with the $N = 2$ case.

5.1 $N = 2$

We recall that the two-variable contribution to the four-point function is

$$\langle QQQQ \rangle_2 = (g_{13} g_{24})^Q S_{02} \sum_{p=2}^{Q} G_p(X_1, X_2) Y^{Q-p} \tag{5.1}$$

If we interchange points 1 and 3 we find that the prefactor changes only by a sign while $Z \mapsto (1 - Z)$. If we interchange points 1 and 4 $Z \mapsto Z^{-1}$ while the prefactor is multiplied by $Y^{Q-2}(X_1 X_2)^{-Q-2}$. Now the number of independent function is determined by the behaviour of the sum as a function of Y. To determine this we can ignore the dependence of the Gs on X_1, X_2. The problem is therefore to find the number of independent polynomials in Y of degree $n := Q - 2$, $f(Y)$, such that

$$f(Y) = f(1 - Y) \qquad \text{and} \qquad f(Y) = Y^n f(Y^{-1}) \tag{5.2}$$

Starting from Y we generate a sequence of six polynomials Y, $(1 - Y)$, Y^{n-1}, $(1 - Y)^{n-1}$, $Y(1 - Y)^{n-1}$, $(1 - Y)Y^{n-1}$ provided that $n \geq 5$. For $n \leq 5$ it is easy to see that this procedure generates a basis of polynomials of degree n, so that there is just one invariant in all of these cases. For $n > 5$ we need to consider a second sequence which can be generated from Y^2, and so forth. We therefore conclude that the number of independent functions of two-variables which is needed in the fully crossing symmetric four-point function of charge Q CPOs is $[1/6\,(Q - 1)]$ where the square brackets denote the smallest integer which is greater than or equal to the number inside.

5.2 $N = 4$

For the $N = 4$ case we start from the expression

$$\langle QQQQ \rangle_2 = (g_{13} g_{24})^Q S_{020} \sum_{p=2}^{Q} \sum_{c=0}^{p-2} G_{pc} \left(\frac{Y_1^{c+1} - Y_2^{c+1}}{Y_1 - Y_2} \right) (Y_1 Y_2)^{Q-p} \tag{5.3}$$

Under the interchange of points 1 and 3 the prefactor here is invariant while $Z \mapsto (1 - Z)$, while under the interchange of points 1 and 4 the prefactor is multiplied by $(Y_1Y_2)^{Q-2}(X_1X_2)^{-(Q+2)}$. Now the sum in (5.3) can be rewritten as a polynomial in the variables $Y_1 + Y_2$ and Y_1Y_2 of degree $n = (Q - 2)$, so we again need to decide how many such polynomials there are which are invariant under the transformations generated by the crossing symmetries. To do this it is convenient to consider instead polynomials of degree n in the variables $y : Y_1Y_2$ and $z := (1 - Y_1)(1 - Y_2)$. If we denote such a polynomial by $f(y, z)$ then the invariance conditions are

$$f(y,z) = f(z,y) \qquad \text{and} \qquad f(y,z) = y^n f(y^{-1}, y^{-1}z)\,. \tag{5.4}$$

The basis elements $w^{rs} := y^r z^s$ behave as

$$w^{rs} \mapsto w^{sr} \tag{5.5}$$
$$w^{rs} \mapsto w^{n-(r+s),s} \tag{5.6}$$

under the crossing transformations. These basis elements can be related to each other in various chains under these transformations. For example,

$$1 \mapsto y^n \mapsto z^n \tag{5.7}$$

and

$$y \mapsto y^{n-1} \mapsto z^{n-1} \mapsto yz^{n-1} \mapsto y^{n-1}z \mapsto z \tag{5.8}$$

In fact, the chains of monomials generated in this fashion either have length 3 or length 6 apart from some exceptional chains of length 1 which occur when $r = s = n/3$.

Using this information one can show that the number, N_Q, of invariant polynomials of this type is given by the following formula:

$$\begin{aligned}
Q-2 &= 0 \bmod 6: & N_Q &= \frac{(Q-2)(Q+4)}{12} + 1 \\
Q-2 &= 1 \bmod 6: & N_Q &= \frac{(Q+3)(Q-1)}{12} \\
Q-2 &= 2 \bmod 6: & N_Q &= \frac{(Q+2)Q}{12} \\
Q-2 &= 3 \bmod 6: & N_Q &= \frac{(Q+1)2}{12} \\
Q-2 &= 4 \bmod 6: & N_Q &= \frac{Q(Q+2)}{12} \\
Q-2 &= 5 \bmod 6: & N_Q &= \frac{(Q-1)(Q+3)}{12}
\end{aligned} \tag{5.9}$$

6. Free four-point functions

As we have stated above, it is expected that the single-variable functions that occur in the four-point functions will take the same forms as they do in the free theory. In this section we express these free functions in terms of the variables we have been using for the case of charge 2 operators.

The free four-point function of four charge two operators, in both $N=2$ and $N=4$, has the form

$$\langle 2222\rangle = A(g_{12}^2g_{34}^2+g_{13}^2g_{24}^2+g_{14}^2g_{23}^2)+B(g_{12}g_{34}g_{13}g_{24}+g_{12}g_{34}g_{14}g_{23}+g_{13}g_{24}g_{14}g_{23}) \quad (6.1)$$

which can be rewritten as

$$\begin{aligned}\langle 2222\rangle = g_{13}^2g_{24}^2\Big(& A\left(1+\operatorname{sdet}(Z)^{-2}+\operatorname{sdet}(1-Z)^{-2}\right)+\\ & +B\left(\operatorname{sdet}(Z)^{-1}+\operatorname{sdet}(1-Z)^{-1}+\operatorname{sdet}(Z)^{-1}\operatorname{sdet}(1-Z)^{-1}\right)\Big) .\end{aligned} \quad (6.2)$$

One must now distinguish between $N=2$ and $N=4$. For $N=2$ one can show that

$$\operatorname{sdet}(1-Z)^{-2} = \sum_{b=1}^{\infty}\sum_{a=0}^{\infty}(a+1)S_{a,b+1}+\sum_{a=0}^{\infty}(a+1)S_{a-1} \quad (6.3)$$

$$\operatorname{sdet}(1-Z)^{-1} = \sum_{a=0}^{\infty}S_{a-1} \quad (6.4)$$

and by inserting these expressions into (6.2) one can rewrite this in the form (4.11) with

$$G(X_1,X_2) = \frac{A}{(1-X_1)^2(1-X_2)^2} \quad (6.5)$$

$$f(X) = A+A\frac{1}{(1-X)^2}+B\frac{1}{(1-X)} \quad (6.6)$$

$$f'(X) = \frac{A}{X}+B\left(1+\frac{1}{1-X}\right). \quad (6.7)$$

Then using the invariance following from (4.15) we can absorb f' into f and G to give

$$G(X_1,X_2) = A\left(\frac{X_1-X_2}{(1-X_1)^2(1-X_2)^2}+\frac{1}{X_1^2X_2^2}\right)-\frac{B}{X_1X_2(1-X_1)(1-X_2)} \quad (6.8)$$

$$f(X) = A\left(\frac{1}{(1-X)^2}+\frac{1}{X^2}+1\right)+B\left(\frac{1}{(1-X)}+\frac{1}{X}+\frac{1}{X(1-X)}\right) \quad (6.9)$$

$$f'(X) = 0\,. \quad (6.10)$$

In this form the crossing symmetry is more apparent.

In $N=4$ we have

$$\operatorname{sdet}(1-Z)^{-2} = \sum_{b=2}^{\infty}\sum_{a=0}^{\infty}(a+1)S_{ab0}+\sum_{a=0}^{\infty}\left((a+1)S_{a-1,1}+(a+1)S_{a-2,0}\right) \quad (6.11)$$

$$\operatorname{sdet}(1-Z)^{-1} = \sum_{a=0}^{\infty}S_{a-2,0} \quad (6.12)$$

and so (6.2) can be rewritten in terms of the functions G_0, f, g, h and the constant C as

$$\langle TTTT\rangle = g_{13}^2g_{24}^2\left(\mathcal{G}_{ab0}[G_0]+\mathcal{F}_1[f]+\mathcal{F}_0[g]+\left(\frac{Y_1Y_2}{X_1X_2}\right)\mathcal{F}_0[h]+C\right) \quad (6.13)$$

where

$$
\begin{aligned}
G_0 &= \frac{A}{(1-X_1)^2(1-X_2)^2} && (6.14)\\
f(X) &= \frac{A}{X(1-X)^2} && (6.15)\\
g(X) &= \frac{-(2A+2B)}{X^2} + \frac{A}{X^2(1-X)^2} + \frac{B}{X(1-X)} && (6.16)\\
h(X) &= \frac{-A}{X^2} + \frac{B}{X(1-X)} && (6.17)\\
C &= 3A+3B\,. && (6.18)
\end{aligned}
$$

In a similar fashion to the $N=2$ case we can absorb the function h to rewrite this in the form

$$
\begin{aligned}
G_0 &= A\left(\frac{1}{(1-X_1)^2(1-X_2)^2} + \frac{1}{(X_1X_2)^2}\right) + \frac{B}{(1-X_1)(1-X_2)X_1X_2} && (6.19)\\
f(X) &= \frac{1}{X}\left(A\left(\frac{1}{(1-X)^2} + \frac{1}{X^2}\right) - \frac{B}{X(1-X)}\right) && (6.20)\\
g(X) &= \frac{-(2A+2B)}{X^2} + \frac{A}{X^2(1-X)^2} + \frac{B}{X(1-X)} && (6.21)\\
h(X) &= 0 && (6.22)\\
C &= 3A+3B\,. && (6.23)
\end{aligned}
$$

7. Conclusions

We have considered in this paper the analysis of four-point superconformal invariants by means of the Schur polynomials. The constraints imposed on the four-point functions by superconformal symmetry come from the Ward identities together with analyticity in the internal coordinates. These conditions can be solved straightforwardly in terms of a number of functions of two variables and a number of single-variable functions, the particular numbers depending on the charge of the operators. The single-variable functions are due to the contributions of protected operators in the OPE and are therefore expected to take their free theory forms as they arise from three-point functions of two CPOs and the protected operator in question. This is partial non-renormalisation.The fact that these three-point functions are themselves non-renormalised can be proved by means of the reduction formula which relates an n-point to an $(n+1)$-point function with an integrated insertion of the stress-energy tensor. It therefore appears that the structure of such four-point functions in $N=4$ SYM is a corollary of the existence of protected operators and the non-renormalisation theorem for three-point functions of them.

We have concentrated in this paper on four-point functions of identical operators but it should be possible to generalise the theory to four-point functions of arbitrary CPOs. It would be interesting to extend the formalism further to include other series C operators, for example, 1/4 BPS superfields. Although this would be technically more complicated, one might expect to find similar features because the short operators in the OPE would again give non-renormalised contributions.

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Although we have focused on the $N = 4$ theory, the formalism is applicable to $N = 2$ and even $N = 0$ as we have seen. We expect that it can be applied to other cases such as the $(2, 0)D = 6$ superconformal theory [34], which also has a harmonic superspace formulation [35]. Indeed, the four-point function of supercurrents has been investigated and compared with the relevant supergravity in [36]. The results are very similar to the four-dimensional case although they were obtained without the use of the reduction formula.

Acknowledgments

This research was supported in part by PPARC SPG grant 613, PPARC Rolling Grant 451 and EU grant HPRN 2000-00122. P. Heslop acknowledges financial support from the RTN European Program HPRN-CT-2000-00148 "Physics Across the Present Energy Frontier: Probing the Origin of Mass."

We are grateful to Emery Sokatchev for many interesting discussions. The results for the number of independent functions of two variables and the partial non-renormalisation of $N = 4$ four-point functions presented here have also been obtained (for the leading component fields) in the CPO multiplets by the use of the reduction formula in superspace and also in a supergravity calculation, the two calculations being in agreement with each other. These results will appear in a forthcoming paper [37].

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